National Mathematical Olympiad of Singapore

SPECIAL ROUND

Date: 25 JULY 2017

Time Given: 1 hour 30 minutes

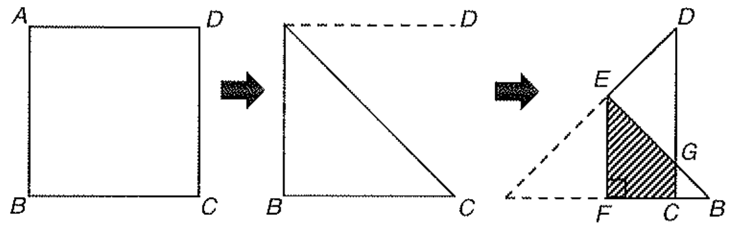
Instructions to Candidates

1. Do not open the booklet until you are told to do so.
2. Answer as many questions as you can.
3. Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.
4. No steps are needed to justify your answers.
5. Questions 1 – 10 are worth 4 marks each.
6. Questions 11 – 20 are worth 5 marks each.
7. No marks will be deducted for wrong answers.
8. Unanswered questions will not get any marks.
9. No calculators or mathematical instruments are allowed.

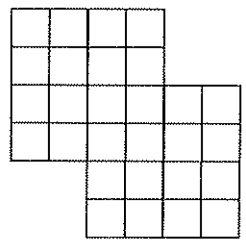
Questions 1 to 10 are worth 4 marks each.

1. Benjamin had 30% more beads than Chloe and 50% fewer beads than Annie. After Annie and Benjamin gave 90 beads and 95 beads respectively to Chloe, Benjamin had $\frac{1}{3}$ as many beads as Annie. How many beads did Chloe have at first?

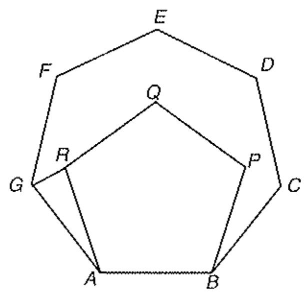
2. The figure at the left below shows a square $ABCD$ with side length 20 cm. When it is first folded along the diagonal BD, a triangle BCD is obtained. Then the triangle BCD is folded along the vertical line EF with side length 12 cm to obtain the figure at the right below. Find the area, in cm^2, of the shaded region in the final figure.



3. How many squares are there in the following figure?



4. As shown in the following figure, $ABCDEFG$ is a regular heptagon and $ABPQR$ is a regular pentagon. Given that $\angle AGR = x^{\circ}$, find the value of $7x$.



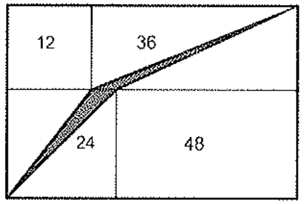
5. There were some oranges and apples at a fruit stall. The ratio of the number of oranges to that of apples was $5:3$. In the morning, $\frac{3}{7}$ of the oranges and some apples were sold. The ratio of the number of oranges to that of apples became $10:7$. The fruit seller bought 60 oranges and 240 apples in the afternoon. In the end, the number of oranges left was the same as the number of apples left. How many apples were there at the fruit stall at first?

6. For the sum of $1+2+3+4+\cdots+n$, it is known that the ones digit is 3 and the tens digit is 0, but the hundreds digit is not 0. Find the smallest possible value of n.

7. George and Heidi were jogging at different uniform speeds on a route AB towards each other from points A and B respectively. When they first met each other, they were 64 m away from point A. When they each reached points B and A respectively, they turned around and jogged back towards their starting points. They met each other the second time at a distance 168 m away from point A. What was the distance, in m, of the jogging route AB?

8. In a competition, Team A and Team B are assigned to complete a construction project respectively. It is known that Team A would take 12 sunny days and Team B would take 15 sunny days to complete the project. However, during rainy days, Team A would complete 40% less per day, and Team B would complete 10% less per day. Both teams start working on the same day and complete the project at the same time a few days later. Find the number of rainy days.

9. The big rectangle below is divided into 4 small rectangles with areas 12 cm^2, 24 cm^2, 36 cm^2, and 48 cm^2 respectively. The lengths of the sides of all rectangles are integers (in cm). Find the area, in cm^2, of the shaded region.



10. Each of the letters ***A, B, C, D, E, F, G, H, I*** and ***J*** represents a different one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 such that

$$\begin{array}{r} \boldsymbol{AB} \\ \boldsymbol{CD} \\ +\ \boldsymbol{EFG} \\ \hline \boldsymbol{HIJ} \end{array}$$

Note that ***A, C, E*** and ***H*** cannot be 0.

If M is the largest possible value of the 3-digit number $\overline{HIJ}$, find the value of M.

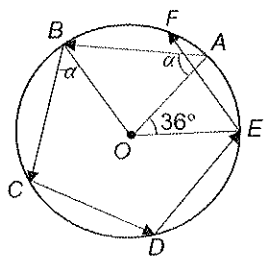
Remark: The following shows one possible arrangement.

$$\begin{array}{r} 98 \\ 76 \\ +\ 231 \\ \hline 405 \end{array}$$

Questions 11 to 20 are worth 5 marks each.

11. Justin, Kaden and Leon run a 300-m race at different uniform speeds. When Justin finishes the race, Kaden is 12 m behind Justin and Leon is 48 m behind Kaden. Kaden has to run for another 2 seconds when Justin finishes the race. How many more seconds does Leon have to run when Kaden finishes the race?

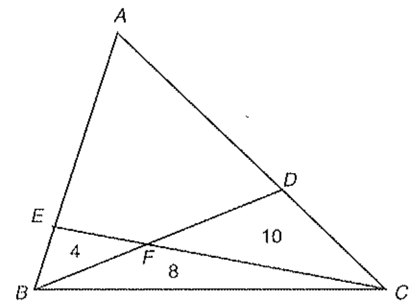
12. Adrian walked on a circular region from starting point A. He walked along the path which formed angle α with radius OA and reached point B on the circle. Then he walked along the path which formed angle α with radius OB and reached point C on the circle. By walking this way, when he reached the point F on the circle, the angle AOE is $36°$. From the point F, find the minimum number of times that he reached a point on the circle before he got to point A again.



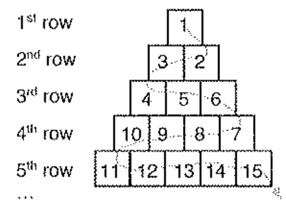
13. There are 5 different whole numbers. The average of some 4 of these 5 numbers is 36, 38, 39, 45 and 49. What is the largest number among these 5 numbers?

14. Within one hour, there are two instances between 11 o'clock and 12 o'clock when the minute hand and the hour hand of the clock make an angle of 70 degrees. What is the time difference, in minutes, between these two timings?

15. In the diagram below, ABC is a triangle. D and E are points on AC and AB respectively. The segments BD and CE intersect at F. If the area of triangles BEF, BCF and CDF are 4 cm^2, 8 cm^2 and 10 cm^2 respectively, find the area, in cm^2, of the quadrilateral $AEFD$.



16. Whole numbers are arranged in the following manner:



For example, the number 10 is placed at 4th row, 1st entry while the number 14 is placed at 5th row, 4th entry.

It is known that the number 2017 is placed at M^{th} row, N^{th} entry. Find the value of $M+N$.

17. Given that r and s are whole numbers such that $\frac{3}{10} < \frac{r}{s} < \frac{5}{16}$, find the smallest value of s.

18. Given that the 7-digit number $\overline{abc2017}$ is a multiple of 13, find the smallest value of the 3-digit number $\overline{abc}$.

19. 5 students A, B, C, D and E went to visit their form teacher Ms Fan on Teachers' Day. Before that, they were trying to guess the 3-digit block number of Ms Fan's place and made the following statements.

Student A: The number is a multiple of 27.
Student B: The number is a multiple of 7.
Student C: The sum of the digits of the number is 17.
Student D: The number is a perfect square.
Student E: The number is a factor of 89100.

Ms Fan told them that only three of these statements are true. What is the 3-digit block number of Ms Fan's place?

20. In the following Figures (a) and (b), each number inside a small triangle is the sum of the numbers inside in the neighbouring small circles. The number inside each circle is either 1, 2, 3, 4, 5, 6, 7 or 8.

The sum of whole numbers inside the circles in Figure (a) is $1+8+8+2+3+5=27$. What is the largest possible sum of whole numbers inside the circles in Figure (b)?

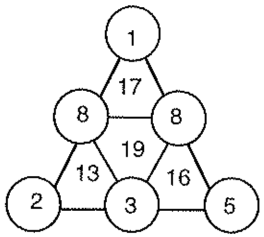


Figure (a)

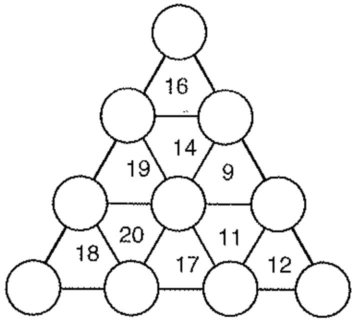


Figure (b)

SPECIAL ROUND – FINAL ANSWER

1.	150	**11.**	10
2.	64	**12.**	35
3.	55	**13.**	63
4.	558	**14.**	40
5.	630	**15.**	50
6.	37	**16.**	128
7.	180	**17.**	13
8.	10	**18.**	112
9.	5	**19.**	324
10.	963	**20.**	52

Remark: To access the full solutions for Special Round questions, scan the following QR code or refer to NUS High School NMOS website at http://nmos.nushigh.edu.sg.

1. Since Benjamin had 30% more beads than Chloe, the ratio of the beads they have is $B:C=13:10$.

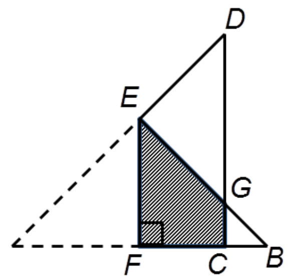
Since Benjamin had 50% fewer beads than Annie, the ratio of the beads they have is $B:A=1:2$.
Hence, the ratio of the beads is $A:B:C=26:13:10$.

	Annie	Benjamin	Chloe
Before	$26u$	$13u$	$10u$
After	$26u-90$	$13u-95$	$10u+90+95=10u+185$

From the condition given, $3(13u-95)=26u-90 \Rightarrow u=15$.
Thus, Chloe had 150 beads at first.

2. Since triangle EFB is a right-angled isosceles triangle, $EF=FB=12$ cm.

Note that $FC+FB=CD=20$ cm, hence $FC=20-12=8$ cm and $CG=CB=12-8=4$ cm.



Finally, area of the shaded region $=\dfrac{1}{2}\times(4+12)\times 8=64$ cm^2.

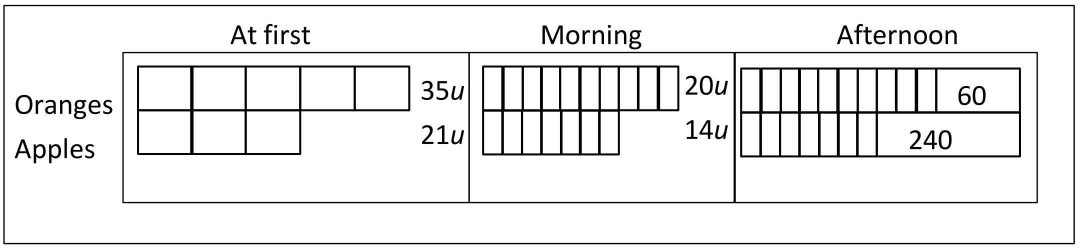
3. The number of 1×1 squares is 28; the number of 2×2 squares is 17; the number of 3×3 squares is 8 and finally the number of 4×4 squares is 2. This means that there are $28+17+8+2=55$ squares.

4. We know that $\angle GAB=\dfrac{5(180^\circ)}{7}$ and $\angle RAB=\dfrac{3(180^\circ)}{5}$, then

$\angle GAR=\angle GAB-\angle RAB=\dfrac{5(180^\circ)}{7}-\dfrac{3(180^\circ)}{5}=\dfrac{144^\circ}{7}$.

Now $\angle AGR=\dfrac{180^\circ-\dfrac{144^\circ}{7}}{2}=\dfrac{558^\circ}{7}=\dfrac{x^\circ}{7}$. This implies that $x=558$.

5.



Note that $\frac{5}{3}=\frac{35}{21}$. We may suppose that there were 35u oranges and 20u apples at first. From the information, there were $35u\times\frac{4}{7}=20u$ oranges and 14u apples in the morning. Finally, there were $20u+60$ oranges and $14u+240$ apples in the afternoon. Since the ratio of the fruits was 1:1 in the afternoon, we have $20u+60=14u+240$, which implies that $u=30$. Hence there were $21u=630$ apples at first.

6. Let the last digit of $1+2+3+4+\cdots+n$ be d. The values of d are shown as follows:

n	1	2	3	4	5	6	7	8	9	10
d	1	**3**	6	0	5	1	8	6	5	5

n	11	12	13	14	15	16	17	18	19	20
d	6	8	1	5	0	6	**3**	1	0	0

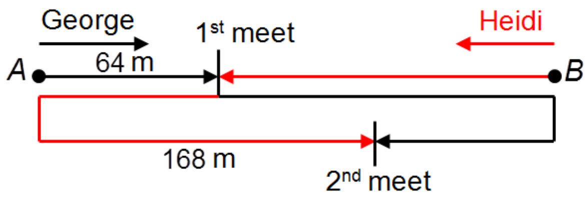
Notice that the possible values of d repeat after every 20 numbers. Hence the last digit of $1+2+3+4+\cdots+n$ is 3 when $n=2,17,22,37,\cdots$.

Note that $1+2=3$, $1+2+\cdots+17=\frac{17\times18}{2}=153$,

$1+2+\cdots+22=\frac{22\times23}{2}=253$ and $1+2+\cdots+37=\frac{37\times38}{2}=703$.

Hence, the smallest possible value of n is 37.

7.



From the 1st meeting point to the 2nd meeting point, Heidi has travelled $64+168=232$ m.

	Distance travelled by George	Distance travelled by Heidi	Total distance travelled by both
Period I: Start $\rightarrow$ 1st meet	64 m		Distance of route *AB*.
Period II: 1st meet $\rightarrow$ 2nd meet		232 m	Double the distance of route *AB*.

$\times 2$

Due to uniform speed, the ratio of travelling distance for Heidi in Period I and II should also be $1:2$. Hence, Heidi travelled $\frac{232}{2}=116$ m when she first met George.

The total distance of route *AB* is $64+116=180$ m.

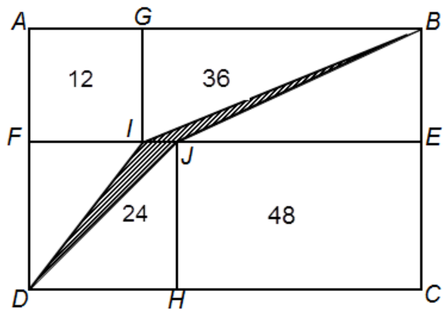
8. During sunny days, team A and B would complete $\frac{1}{12}$ and $\frac{1}{15}$ of the project respectively. Team A would complete $\frac{1}{12}-\frac{1}{15}=\frac{1}{60}$ more of the job.

During rainy days, team A would complete $\frac{1}{12}\times(1-40\%)=\frac{1}{20}$ of the project per day, and team B would complete $\frac{1}{15}\times(1-10\%)=\frac{3}{50}$ of the project per day. Team B would complete $\frac{3}{50}-\frac{1}{20}=\frac{1}{100}$ more of the job.

Suppose altogether there are m sunny days and n rainy days, since both teams complete the project on the same day, $\frac{1}{60}\times m=\frac{1}{100}\times n$. This means that $m:n=\frac{1}{100}:\frac{1}{60}=3:5$.

If $m=3, n=5$, team A would only complete $\frac{1}{12}\times 3+\frac{1}{20}\times 5=\frac{1}{2}$ of the project. Therefore, there are 10 rainy days altogether.

9. Refer to the diagram below. Let $x = FI$, $y = IJ$ and $z = JE$.



As the ratio of the bases of rectangles is the same as the ratio of the areas of rectangles, we have

$$\begin{cases} x:(y+z) = 12:36 = 1:3 = 3:9 \\ (x+y):z = 24:48 = 1:2 = 4:8 \end{cases}.$$

This means that $x:y:z = 3:1:8$.

Note that the area of triangle *IJB*: the area of rectangle *AGFI* is 1:6, and so the area of triangle *IJB* is $\frac{1}{6}(12) = 2$ cm^2.

Next, the area of triangle *IJD*: the area of rectangle *DHJF* is 1:8, and so the area of triangle *IJD* is $\frac{1}{8}(24) = 3$ cm^2.

Finally, the area of the shaded region is $2 + 3 = 5$ cm^2.

10. We claim that $M = 963$ and one possible arrangement is as follows.

$$\begin{array}{r} 4\ 1 \\ 5\ 2 \\ +\ \ 8\ 7\ 0 \\ \hline 9\ 6\ 3 \\ \hline \end{array}$$

In order to obtain M, we have $H = 9$.

$$\begin{array}{r} \boldsymbol{A}\ \boldsymbol{B} \\ \boldsymbol{C}\ \boldsymbol{D} \\ +\ \ \boldsymbol{E}\ \boldsymbol{F}\ \boldsymbol{G} \\ \hline 9\ \boldsymbol{I}\ \boldsymbol{J} \\ \hline \end{array} \qquad \begin{array}{r} \boldsymbol{A}\ \boldsymbol{B} \\ \boldsymbol{C}\ \boldsymbol{D} \\ +\ \ 8\ \boldsymbol{F}\ \boldsymbol{G} \\ \hline 9\ \boldsymbol{I}\ \boldsymbol{J} \\ \hline \end{array} \qquad \begin{array}{r} \boldsymbol{A}\ \boldsymbol{B} \\ \boldsymbol{C}\ \boldsymbol{D} \\ +\ \ 8\ \boldsymbol{F}\ \boldsymbol{G} \\ \hline 9\ 6\ \boldsymbol{J} \\ \hline \end{array}$$

Then $E = 8$ or $E = 7$.

Suppose $E = 7$. Then $\overline{AB} + \overline{CD} + \overline{FG} \le 84 + 63 + 52 < 200$ which means that $\overline{AB} + \overline{CD} + \overline{EFG} = 700 + \left(\overline{AB} + \overline{CD} + \overline{FG}\right) < 900$, a contradiction.

Now $E = 8$.

The largest number left is 7.

Suppose $I = 7$. Then $\overline{AB} + \overline{CD} + \overline{FG} \le 63 + 52 + 41 < 170$ which means that $\overline{AB} + \overline{CD} + \overline{EFG} = 800 + \left(\overline{AB} + \overline{CD} + \overline{FG}\right) < 970$, a contradiction.

Now $I = 6$ and as $\overline{HIJ} = 963$ is possible, we need to show that $\overline{HIJ} \ne 967$, 965 or 964.

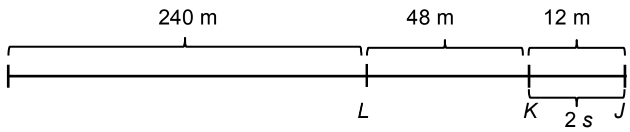
Since $A + B + C + D + F + G + J = 0 + 1 + 2 + 3 + 4 + 5 + 7 = 22$ and, $\overline{AB} + \overline{CD} + \overline{FG} = 10\left(A + C + F\right) + \left(B + D + G\right) = 9\left(A + C + F\right) + \left(22 - J\right)$ and

$\overline{AB} + \overline{CD} + \overline{FG} = 160 + J$, this means that $9\left(A + C + F\right) = 138 + 2J$.

Note that $138 + 2J$ is not a multiple of 9 if $J = 7$, 5 or 4.

This completes the proof.

11.



Suppose Justin spent t seconds to finish the race. Then Justin's speed is $\frac{300}{t}$ m s^{-1}. This implies that Kaden's and Leon's speeds are $\frac{288}{t}$ m s^{-1} and $\frac{240}{t}$ m s^{-1} respectively. From the condition, it is known that Kaden's speed is $\frac{12}{2} = 6$ m s^{-1} and hence $\frac{288}{t} = 6 \Rightarrow t = 48$ s.

Now Leon's speed is $\frac{240}{48} = 5$ m s^{-1} and he has to run for another $\frac{60}{5} - 2 = 10$ s when Kaden finishes the race.

12. Since $\angle AOB = \angle BOC = \angle COD = \angle DOE$, every time Adrian moved to the next point on the circle, he rotated $\frac{360° - 36°}{4} = 81°$ relative to centre O.

Before Adrian reached point A for the second time, the total angle that rotated relative to centre O should be a common multiple of $81°$ and $360°$. The least common multiple of 81 and 360 is 40×81. Hence, from the point F, Adrian reached at least $40 - 5 = 35$ points before he got to the point A for the second time.

13. Let a, b, c, d and e be the 5 whole numbers such that $a < b < c < d < e$. Now we have $\frac{a+b+c+d}{4} = 36$, $\frac{a+b+c+e}{4} = 38$, $\frac{a+b+d+e}{4} = 39$, $\frac{a+c+d+e}{4} = 45$ and $\frac{b+c+d+e}{4} = 49$. Take the sum of these equations, we obtain $a+b+c+d+e = 36+38+39+45+49 = 207$. Now the largest whole number among these 5 numbers is e, which is $(a+b+c+d+e)-(a+b+c+d) = 207 - 4(36) = 63$.

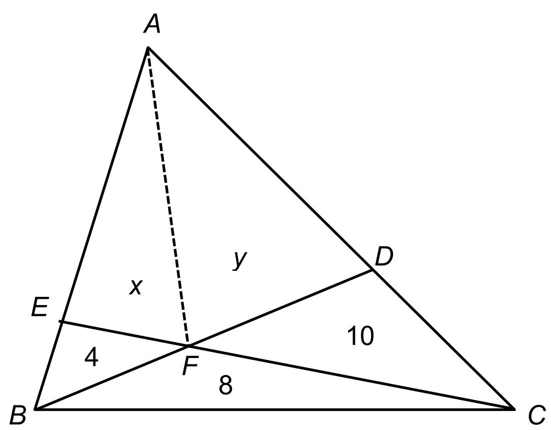
14. In every minute, the minute hand moves $\frac{1}{60} \times 360° = 6°$ while the hour hand moves $\frac{1}{60} \times \frac{360°}{12} = 0.5°$. At 11 o'clock, the angle between the minute hand and the hour hand is $30°$. Then x minutes after 11 o'clock, the angle between the minute hand and the hour hand is $30° + 5.5x°$.

Suppose $x = x_1$ and $x = x_2$ are when the hands make an angle of 70 degrees. Then $\begin{cases} 30° + 5.5x_1° = 70° \\ 30° + 5.5x_2° = 290° \end{cases}$. The difference of the two equations give $5.5(x_2 - x_1) = 220$, which means that the time difference is $x_2 - x_1 = \frac{220}{5.5} = 40$.

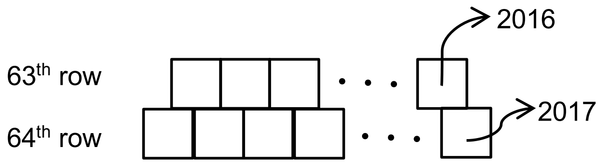
15. Join the points A and F by a line segment. Let the area of triangle AFE be x cm^2 and the area of triangle AFD be y cm^2. The area of the quadrilateral $AEFD$ is $(x + y)$ cm^2. For two triangles with the same height, we know that the ratio of their areas is equal to the ratio of their bases.

Thus $\frac{4+x}{y} = \frac{8}{10} = \frac{4}{5}$ and $\frac{x}{y+10} = \frac{4}{8} = \frac{1}{2}$. From the second equation, $y = 2x - 10$, and from the first equation, $20 + 5x = 4y = 4(2x - 10)$, which means that $x = \frac{60}{3} = 20$ and $y = 2x - 10 = 2(20) - 10 = 30$.

Now $x + y = 50$.



16. If k is odd, then the number placed in the k^{th} row, k^{th} entry is $\dfrac{k(k+1)}{2}$. Note that if $k = 63$, the value of $\dfrac{k(k+1)}{2}$ is $\dfrac{63(64)}{2} = 2016$.



Therefore 2017 is placed at 64th row and 64th entry. So $M = N = 64$ and thus $M + N = 128$.

17. As $\dfrac{3}{10} < \dfrac{r}{s} < \dfrac{5}{16}$, we have $\dfrac{16r}{5} < s < \dfrac{10r}{3}$.

If $r = 1$, then $3.2 = \dfrac{16}{5} < s < \dfrac{10}{3} = 3.33$, so s is not a whole number.

If $r = 2$, then $6.4 = \dfrac{32}{5} < s < \dfrac{20}{3} = 6.67$, so s is not a whole number.

If $r = 3$, then $9.6 = \dfrac{48}{5} < s < \dfrac{30}{3} = 10$, so s is not a whole number.

If $r = 4$, then $12.8 = \dfrac{64}{5} < s < \dfrac{40}{3} = 13.33$, so $s = 13$.

18. Suppose $13 \times N = \overline{abc2017}$. Then the last digit of N is 9.

Now let the second last digit of N be p. Note that $13 \times 9 = 117$. This means $13p$ ends in 0. So p is 0 (see Figure (a))

Next, let the third last digit of N be q. This means $13q$ ends in 9. So q is 3. (see Figure (b))

Next, let the 4^{th} last digit of N be r. Note that $13 \times 3 = 39$. This means $13r$ ends in 8. So r is 6. (see Figure (c))

Finally, let the 5^{th} last digit of N be s. Note that $13 \times 6 = 78$. This means $13s + 7 > 100$. So s is 8. (see Figure (d))

Now the smallest value of $\overline{abc}$ is 112. (see Figure (d))

$$\begin{array}{rr} & 13 \\ \text{X} & {*}{*}{*}{*}p9 \\ \hline & 117 \\ & \\ \hline & abc2017 \\ \hline \end{array}$$

Figure (a)

$$\begin{array}{rr} & 13 \\ \text{X} & {*}{*}{*}q09 \\ \hline & 117 \\ & {*}{*}{*} \\ \hline & abc2017 \\ \hline \end{array}$$

Figure (b)

$$\begin{array}{rr} & 13 \\ \text{X} & {*}{*}r309 \\ \hline & 117 \\ & 39 \\ & {*}{*}{*} \\ \hline & abc2017 \\ \hline \end{array}$$

Figure (c)

$$\begin{array}{rr} & 13 \\ \text{X} & 86309 \\ \hline & 117 \\ & 39 \\ & 78 \\ & 104 \\ \hline & abc2017 \end{array}$$

Figure (d)

19. If statement *A* is true, then the sum of the digits of the block number should be a multiple of 9 and so statement *C* must be false. Hence statement *A* and *C* does not hold at the same time.

Since $89100 = 2^2 \times 3^4 \times 5^2 \times 11$, if the number is a factor of 89100, it should not be a multiple of 7. Hence statement *B* and *E* does not hold at the same time.

This implies that statement *D* is true. Hence the 3-digit number is k^2 for some $k = 11, 12, \cdots, 31$.

Now we are left with 4 possible cases:

(i) Statements *B*, *A* and *D* are true.

(ii) Statements *B*, *C* and *D* are true.

(iii) Statements *E*, *C* and *D* are true.

(iv) Statements *E*, *A* and *D* are true.

If statement *B* is true, then the 3-digit number is $(7m)^2$ for some $m = 2, 3, 4$, which means that $14^2 = 196$, $21^2 = 441$ and $28^2 = 784$ are the possible 3-digit number. None of these make statements *A* or *C* true.

This means that statement *E* is true. We are left with 2 possible cases:

(iii) Statements *E*, *C* and *D* are true.

(iv) Statements *E*, *A* and *D* are true.

Since statements *E* and *D* are true, the 3-digit number is $\left(2^r \times 5^s \times 3^t\right)^2$ for some $r, s = 0, 1$ and some $t = 0, 1, 2$.

If the sum of the digits of the 3-digit number is not a multiple of 9, then the 3-digit number does not contain a factor of 9, which means that it must be $(2 \times 5)^2 = 100$, which means the sum of the digits is 1. Hence statement *C* is not true.

Finally, statements *E*, *A* and *D* are true and the 3-digit number is $2^2 \times 3^4 = 324$.

20. The sum is 52.

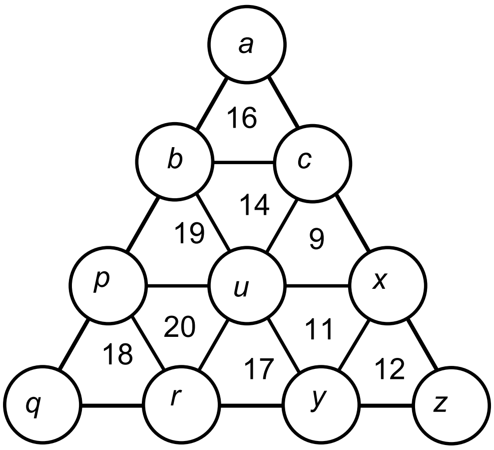


Figure (A)

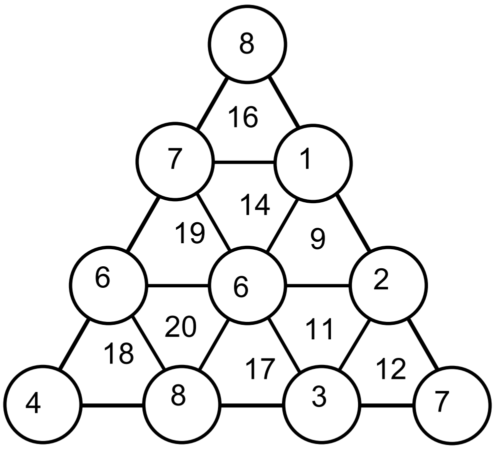


Figure (B)

Refer to the Figure (A) above, the sum is

$$(a+b+c)+(p+q+r)+(x+y+z)+u.$$

We know that $a+b+c=16$, $p+q+r=18$ and $x+y+z=12$. Hence the sum is $16+18+12+u=46+u$.

Note that $a+b+c=16$ and $u+b+c=14$. The difference of the equations gives $u=a-2\leq 8-2=6$. Now the desired sum is not more than 52.

The Figure (B) above shows a possible solution when the sum is 52. So the largest possible sum is 52.